

AN EXAMPLE OF  
GENERALIZED SCHUR OPERATORS  
INVOLVING PLANAR BINARY TREES

NUMATA YASUhide

ABSTRACT. Young's lattice is a prototypical example of differential posets. Differential posets have the Robinson correspondence, the correspondence between permutations and pairs of standard tableaux with the same shape, as in the case of Young's lattice. Fomin introduced generalized Schur operators to generalize the method of Robinson correspondence in differential posets to the Robinson-Schensted-Knuth correspondence, the correspondence between certain matrices and pairs of semi-standard tableaux with the same shape. In this paper, we introduce operators on the vector space whose basis is the set of planar binary trees. To prove that the operators are generalized Schur operators, we construct a correspondence, which is an extension of Fomin's  $r$ -correspondence for them.

1. INTRODUCTION

Stanley introduced differential posets in [11, 12]. Young's lattice is a prototypical example of differential posets. Young's lattice has the Robinson correspondence, the correspondence between permutations and pairs of standard tableaux whose shapes are the same Young diagram. This correspondence was generalized for differential posets or dual graphs (generalizations of differential posets [2]) by Fomin [1, 3]. (See also [10].)

Young's lattice also has the Robinson-Schensted-Knuth correspondence, the correspondence between certain matrices and pairs of semi-standard tableaux. Fomin [4] introduced operators called generalized Schur operators, and generalized the method of the Robinson correspondence to that of the Robinson-Schensted-Knuth correspondence.

In this paper, we introduce linear operators on the vector space whose basis is the set of binary trees. We show the operators are generalized Schur operators. To prove this, we construct an extension of an  $r$ -correspondence.

2. GENERALIZED SCHUR OPERATORS

In this section, we recall generalized Schur operators introduced by Fomin [4].

Let  $K$  be a field of characteristic zero that contains all formal power series of variables  $t, t', t_1, t_2, \dots$ . Let  $V_i$  be finite-dimensional  $K$ -vector spaces for all  $i \in \mathbb{Z}$ . Fix a basis  $Y_i$  of each  $V_i$  so that  $V_i = KY_i$ . Let  $Y = \bigcup_i Y_i$ ,  $V = \bigoplus_i V_i$  and  $\widehat{V} = \prod_i V_i$ , i.e.,  $V$  is the vector space consisting of all finite linear combinations of elements of  $Y$ , and  $\widehat{V}$  is the vector space consisting of all linear combinations of elements of  $Y$ .

For a sequence  $\{A_i\}$  and a formal variable  $x$ , we write  $A(x)$  for the generating function  $\sum_{i \geq 0} A_i x^i$ .

**Definition 2.1.** We call  $D(t_1) \cdots D(t_n)$  and  $U(t_n) \cdots U(t_1)$  *generalized Schur operators with  $\{a_m\}$*  if the following conditions are satisfied:

- $\{a_m\}$  is a sequence of elements of  $K$ .
- $U_i$  is a linear map on  $V$  satisfying  $U_i(V_j) \subset V_{j+i}$  for all  $j$ .
- $D_i$  is a linear map on  $V$  satisfying  $D_i(V_j) \subset V_{j-i}$  for all  $j$ .
- The equation  $D(t')U(t) = a(tt')U(t)D(t')$  holds.

*Remark 2.2.* Let  $\langle \cdot, \cdot \rangle$  be the natural pairing in  $KY$ , i.e., the bilinear form on  $\widehat{V} \times V$  such that  $\langle \sum_{\lambda \in Y} a_\lambda \lambda, \sum_{\mu \in Y} b_\mu \mu \rangle = \sum_{\lambda \in Y} a_\lambda b_\lambda$ . For generalized Schur operators  $D(t_1) \cdots D(t_n)$  and  $U(t_n) \cdots U(t_1)$ ,  $U_i^*$  and  $D_i^*$  denote the maps obtained from the adjoints of  $U_i$  and  $D_i$  with respect to  $\langle \cdot, \cdot \rangle$  by restricting to  $V$ , respectively. For all  $i$ ,  $U_i^*$  and  $D_i^*$  are linear maps on  $V$  satisfying  $U_i^*(V_j) \subset V_{j-i}$  and  $D_i^*(V_j) \subset V_{j+i}$ . By definition,

$$\langle v, U_i w \rangle = \langle w, U_i^* v \rangle, \quad \langle v, D_i w \rangle = \langle w, D_i^* v \rangle$$

for  $v, w \in V$ . We write  $U^*(t)$  and  $D^*(t)$  for  $\sum U_i^* t^i$  and  $\sum D_i^* t^i$ . By definition,

$$\langle U(t)\mu, \lambda \rangle = \langle U^*(t)\lambda, \mu \rangle, \quad \langle D(t)\mu, \lambda \rangle = \langle D^*(t)\lambda, \mu \rangle$$

for  $\lambda, \mu \in Y$ . The equation  $D(t')U(t) = a(tt')U(t)D(t')$  implies the equation  $U^*(t')D^*(t) = a(tt')D^*(t)U^*(t')$ . Hence  $U^*(t_1) \cdots U^*(t_n)$  and  $D^*(t_n) \cdots D^*(t_1)$  are generalized Schur operators with  $\{a_m\}$  when  $D(t_1) \cdots D(t_n)$  and  $U(t_n) \cdots U(t_1)$  are.

### 3. DEFINITION

In this section, first we recall the definition of rooted planar binary trees and labellings on them. Next we introduce linear operators on the vector space whose basis is the set of rooted planar binary trees.

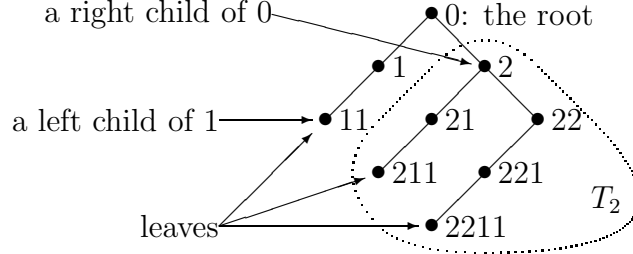
**3.1. Rooted Planar Binary Trees.** We define rooted planar binary trees and their labellings.

Let  $F$  be the monoid of words generated by the alphabet  $\{1, 2\}$  and let  $0$  denote the word whose length is 0. We identify  $F$  with a poset by  $v \leq vw$  for  $v, w \in F$ . We call an ideal of poset  $F$  a *rooted planar binary tree* or shortly *tree*. Let  $\mathbb{T}$  denote the set of trees.

Let  $T$  be a tree. An element of  $T$  is called a *node* of  $T$ . We write  $\mathbb{T}_i$  for the set of trees of  $i$  nodes. We respectively call nodes  $v_2$  and  $v_1$  *right* and *left children* of  $v$ . A node without children is called a *leaf*. If  $T$  is nonempty,  $0 \in T$ . We call  $0$  the *root* of  $T$ .

For  $T \in \mathbb{T}$  and  $v \in F$ , we define  $T_v$  by  $T_v := \{ w \in T \mid v \leq w \}$ .

*Example 3.1.* For a tree  $T = \{ 0, 11, 2, 21, 211, 22, 221, 2211 \}$ , the root, leaves and so on are as follows:



**Definition 3.2.** Let  $T$  be a tree and  $m$  a positive integer. We call a map  $\varphi : T \rightarrow \{ 1, \dots, m \}$  a *right-strictly-increasing labelling* if

- $\varphi(w) \leq \varphi(v)$  for  $w \in T$  and  $v \in T_{w_1}$  and
- $\varphi(w) < \varphi(v)$  for  $w \in T$  and  $v \in T_{w_2}$ .

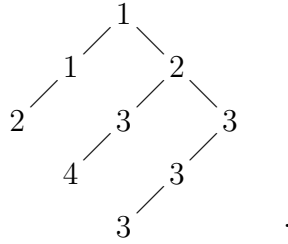
We call a map  $\phi : T \rightarrow \{ 1, \dots, m \}$  a *left-strictly-increasing labelling* if

- $\phi(w) < \phi(v)$  for  $w \in T$  and  $v \in T_{w_1}$  and
- $\phi(w) \leq \phi(v)$  for  $w \in T$  and  $v \in T_{w_2}$ .

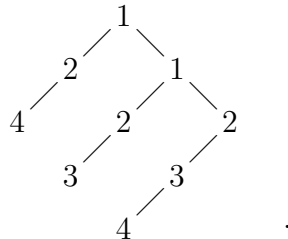
We call a map  $\psi : T \rightarrow \{ 1, \dots, m \}$  a *binary-searching labelling* if

- $\psi(w) \geq \psi(v)$  for  $w \in T$  and  $v \in T_{w_1}$  and
- $\psi(w) < \psi(v)$  for  $w \in T$  and  $v \in T_{w_2}$ .

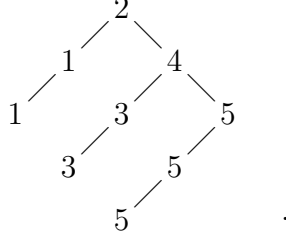
*Example 3.3.* The following is a right-strictly-increasing labelling:



The following is a left-strictly-increasing labelling:



The following is a binary-searching labelling:



**3.2. Definition of our generalized Schur operators.** In this section, we define linear operators  $U_i$ ,  $U'_i$ ,  $D_i$ . In Section 4, we shall show that these linear operators are generalized Schur operators.

**3.2.1. Up operators.** First we define up operators  $U_i$  and consider a relation between the up operators  $U_i$  and right-strictly labellings. Next we define  $U'_i$  and consider a relation between the up operators  $U'_i$  and left-strictly labellings.

**Definition 3.4.** We define the edges  $G_U$  of oriented graphs whose vertices are trees to be the set of pairs  $(T, T')$  of trees satisfying the following:

- $T \subset T'$ .
- For each  $w \in T' \setminus T$ , there exists  $v_w \in T$  such that  $w = v_w 1^n$  or  $w = v_w 21^n$  for some nonnegative integer  $n$  if  $T \neq \emptyset$ .
- For each  $w \in T' \setminus T$ ,  $w = 1^n$  for some nonnegative integer  $n$  if  $T = \emptyset$ .

We call  $T'$  a tree obtained from  $T$  by adding some nodes right-strictly if  $(T, T') \in G_U$ . We define  $G_{U_i}$  by

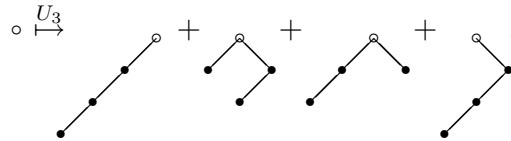
$$G_{U_i} = \{ (T, T') \in G_U \mid |T| + i = |T'| \}.$$

**Definition 3.5.** For  $i \in \mathbb{N}$  and  $T \in \mathbb{T}$ , we define linear operators  $U_i$  on  $K\mathbb{T}$  by

$$U_i T = \sum_{T': (T, T') \in G_{U_i}} T'.$$

Equivalently,  $U_i T$  is the sum of all trees obtained from  $T$  by adding  $i$  nodes right-strictly.

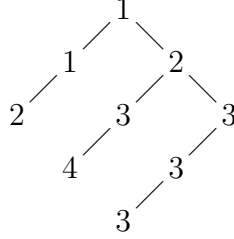
*Example 3.6.* For example,  $U_3$  acts on  $\{0\}$  as follows:



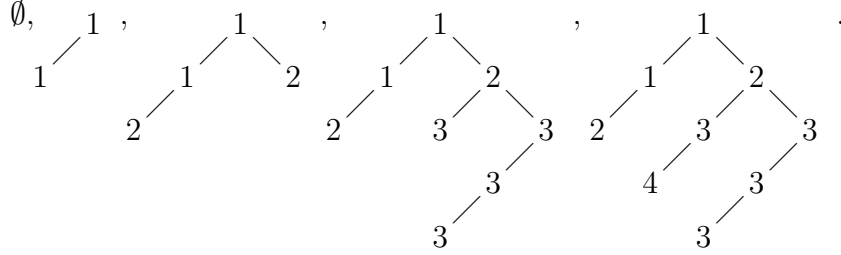
*Remark 3.7.* Let  $\varphi$  be a right-strictly-increasing labelling. The inverse image  $\varphi^{-1}(\{1, \dots, n+1\})$  is the tree obtained from the inverse image  $\varphi^{-1}(\{1, \dots, n\})$  by adding some nodes right-strictly. Hence we identify

right-strictly-increasing labellings with paths  $(\emptyset = T^0, T^1, \dots, T^m)$  of  $G_U$ .

*Example 3.8.* For example, we identify a right-strictly-increasing labelling



with a sequence



Next we define other up operators  $U'_i$ .

**Definition 3.9.** We define the edges  $G_{U'}$  of oriented graphs whose vertices are trees to be the set of pairs  $(T, T')$  of trees satisfying the following:

- $T \subset T'$ .
- For each  $w \in T' \setminus T$ , there exists  $v_w \in T$  such that  $w = v_w 2^n$  or  $w = v_w 12^n$  for some nonnegative integer  $n$  if  $T \neq \emptyset$ .
- For each  $w \in T' \setminus T$ ,  $w = 2^n$  for some nonnegative integer  $n$  if  $T = \emptyset$ .

We call  $T'$  a tree obtained from  $T$  by adding some nodes left-strictly if  $(T, T') \in G_{U'}$ . We define  $G_{U'_i}$  by

$$G_{U'_i} = \{ (T, T') \in G_U \mid |T| + i = |T'| \}.$$

**Definition 3.10.** For  $i \in \mathbb{N}$  and  $T \in \mathbb{T}$ , we define linear operators  $U'_i$  on  $K\mathbb{T}$  to be

$$U'_i T = \sum_{T': (T, T') \in G_{U'_i}} T'.$$

Equivalently,  $U'_i T$  is the sum of all trees obtained from  $T$  by adding  $i$  nodes left-strictly.

*Remark 3.11.* Similarly to the case of  $U_i$  and right-strictly-increasing labellings, we identify left-strictly-increasing labellings with paths  $(\emptyset = T^0, T^1, \dots, T^m)$  of  $G_{U'}$ .

First we prepare some terms to define the linear operators  $D_i$ .

$$T \ominus w = (T \setminus T_w) \cup \{wv \mid w1v \in T_w\}.$$
$$\begin{cases} \nu_{T,w}(wv) = w1v & (wv \in T_w) \\ \nu_{T,w}(v') = v' & (v' \notin T_w). \end{cases}$$

inclusion  $\nu_{T,w}$  maps the nodes in  $\bigcup_{i \in I} \mathcal{N}_i$  of  $T \ominus w$  to the nodes in  $\bigcup_{i \in I} \mathcal{N}_i$  of  $T$ ,

For  $T \in \mathbb{T}$ , let  $E_T$  denote  $\{w \in T \mid \text{If } w = v1w' \text{ then } v2 \notin T\}$ . Roughly speaking, it is the set of nodes of  $T$  between the root 0 and the right-most node of  $T$ . We define  $r_T$  by  $r_T = E_T \cap R_T$ . The set  $r_T$  is a chain. Let  $r_T = \{w_{T,1} < w_{T,2} < w_{T,3} < \dots < w_{T,k}\}$ . Let  $r_{T,i}$  denote the ideal  $\{w_{T,1}, w_{T,2}, w_{T,3}, \dots, w_{T,i}\}$  of  $r_T$  consisting of  $i$  nodes.

where nodes on thick lines are in  $E_T$  and  $\bullet$  are in  $R_T$ .

We define  $T \ominus r_{T,i}$  inductively by

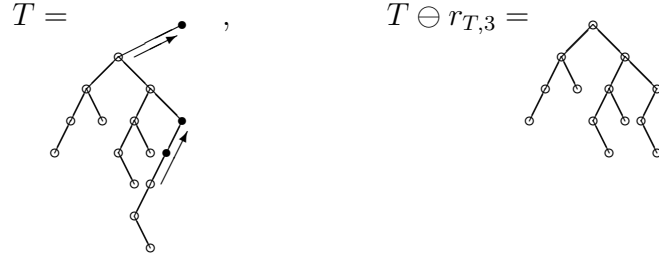
$$\begin{cases} (T \ominus w_{T,i}) \ominus r_{T,i-1} & i > 0 \\ T & i = 0. \end{cases}$$

The natural inclusion  $\nu_{T,w}$  induces the natural inclusion

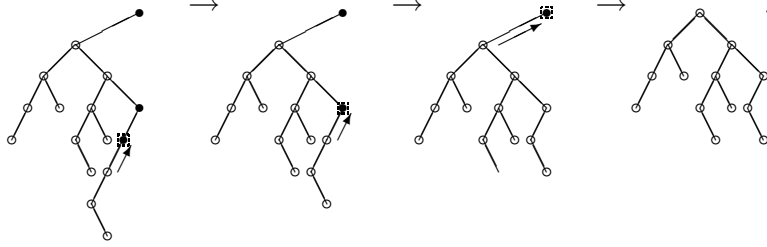
$$\nu_{T,i} = \nu_{T \ominus w_{T,i}, i-1} \circ \nu_{T, w_{T,i}}$$

from  $T \ominus r_{T,i}$  to  $T$ .

*Example 3.14.* For



since



The natural inclusion  $\nu_{T,3}$  maps the nodes  $\circ$  in  $T \ominus r_{T,3}$  to the former nodes  $\circ$  in  $T$ .

We also define a bijection  $\tilde{\nu}_{T,i}$  from the words  $F$  of  $\{1, 2\}$  to  $F \setminus r_{T,i}$

$$\tilde{\nu}_{T,i}(w) = \nu_{T,i}(v)v',$$

where  $w = vv'$  and  $v = \max \{ u \in T \ominus r_{T,i} \mid w = uu' \}$ . By definition,  $\tilde{\nu}_{T,i}(w) = \nu_{T,i}(w)$  for  $w \in T \ominus r_{T,i}$ .

**Definition 3.15.** We define the edges  $G_{D_i}$  and  $G_D$  of graphs whose vertices are trees by

$$G_{D_i} = \{ (T \ominus r_{T,i}, T) \mid |r_T| \geq i \}$$

and

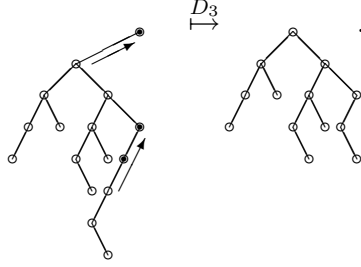
$$G_D = \bigcup_i G_{D_i}.$$

*Remark 3.16.* By definition,  $G_{D_0} = \{ (T, T) \mid T \in \mathbb{T} \}$ . For each  $i$  and each  $T \in \mathbb{T}$ , the in-degree of  $T$  in  $G_{D_i}$ , i.e.  $|\{ (T', T) \in G_{D_i} \}|$ , is 1.

**Definition 3.17.** We define linear operators  $D_i T$  to be  $T'$  such that  $(T', T) \in G_{D_i}$  for  $T \in \mathbb{T}$ .

Roughly speaking,  $D_i T$  is the tree obtained from  $T$  by evacuating the  $i$  topmost nodes without a child on its right between the root 0 and the rightmost leaf of  $T$ .

*Example 3.18.* For example,  $D_3$  acts as follows:



Next we consider a relation between  $G_D$  and binary-searching labellings.

*Remark 3.19.* Let  $\psi_m : T \rightarrow \{1, \dots, m\}$  be a binary-searching labelling. By the definition of binary-searching labelling, the inverse image  $\psi_m^{-1}(\{m\})$  equals  $r_{T, k_m} = \{w_{T, 1}, \dots, w_{T, k_m}\}$  for some  $k_m$ . Hence we can construct  $T \ominus \psi_m^{-1}(\{m\})$ . The natural inclusion  $\nu_{T, k_m}$  induces a binary-searching labelling

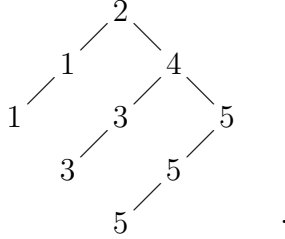
$$\psi_{m-1} = \psi_m \circ \nu_{T,k_m} : T \ominus \psi_m^{-1}(\{m\}) \rightarrow \{1, \dots, m-1\}.$$

Hence we identify binary-searching labellings with paths

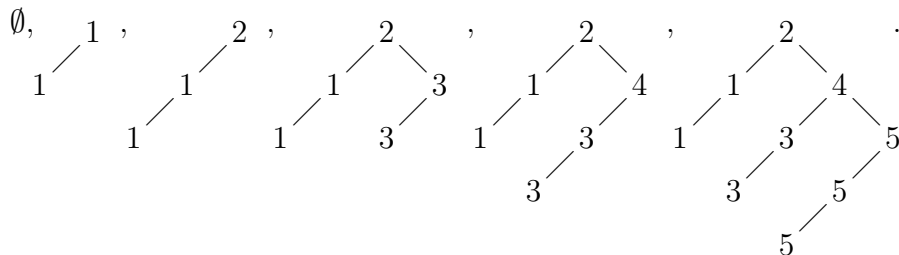
$$(\emptyset = T^0, T^1, \dots, T^m)$$

of  $G_D$ .

*Example 3.20.* For example, we identify a binary-searching labelling



with a sequence



## 4. MAIN RESULTS

In this section, we show that  $D(t_1) \cdots D(t_n)$  and  $U(t_n) \cdots U(t_1)$  are generalized Schur operators with  $\{1, 1, 1, \dots\}$ . We also show that  $D(t_1) \cdots D(t_n)$  and  $U'(t_n) \cdots U'(t_1)$  are generalized Schur operators with  $\{1, 1, 0, 0, 0, 0, \dots\}$ . To prove the assertion for  $D(t_1) \cdots D(t_n)$  and  $U(t_n) \cdots U(t_1)$ , we construct correspondences between  $N_{i,j}(T, T')$  and  $\tilde{S}_{j,i}(T, T')$ , where  $N_{i,j}(T, T')$  is the set of paths of graphs from  $T$  to  $T'$  via  $G_{D_j}$  after  $G_{U_i}$ , and  $\tilde{S}_{j,i}(T, T')$  is the set of paths of graphs from  $T$  to  $T'$  via  $G_{U_{i-k}}$  after  $G_{D_{j-k}}$ , where  $0 \leq k \leq \min\{i, j\}$ . (See Definition 4.9.) The correspondence for  $i = j = 1$  is an  $r$ -correspondence introduced by Fomin [3], which is needed to construct Robinson correspondences for  $r$ -dual graphs. We also prove the assertion for  $D(t_1) \cdots D(t_n)$  and  $U(t_n) \cdots U(t_1)$  similarly.

**4.1. Main Theorems.** We prove the following theorems in Section 4.2.

**Theorem 4.1.** *Let  $D_i$  be the linear operators defined in Definition 3.17 and  $U_i$  the linear operators defined in Definition 3.5. The operators  $D(t)$  and  $U(t')$  satisfy the equation*

$$D(t)U(t') = \frac{1}{1 - tt'} U(t')D(t).$$

*Equivalently, operators  $D(t_1) \cdots D(t_n)$  and  $U(t_n) \cdots U(t_1)$  are generalized Schur operators with  $\{1, 1, 1, \dots\}$ .*

**Theorem 4.2.** *Let  $D_i$  be the linear operators defined in Definition 3.17 and  $U'_i$  the linear operators defined in Definition 3.10. The operators  $D(t)$  and  $U'(t')$  satisfy the equation*

$$D(t)U'(t') = (1 + tt')U'(t')D(t).$$

*Equivalently, operators  $D(t_1) \cdots D(t_n)$  and  $U'(t_n) \cdots U'(t_1)$  are generalized Schur operators with  $\{1, 1, 0, 0, 0, 0, \dots\}$ .*

**Corollary 4.3.** *A pair  $(G_{U_1} = G_{U'_1}, G_{D_1})$  of graded graphs is an example of 1-dual graphs in the sense of Fomin [4]. Equivalently,  $U_1$  and  $D_1$  satisfy the equation*

$$D_1 U_1 - U_1 D_1 = I,$$

*where  $I$  is the identity map on  $V$ .*

**Remark 4.4.** Janvier Nzeutchap [8] constructs  $r$ -dual graphs from dual Hopf algebras. The 1-dual graphs obtained from the Loday-Ronco algebra by his method are  $G_{U_1}$  and  $G_{D_1}$ .

**Corollary 4.5.** *The up and down operators  $(U_1, D)$  of  $(G_{U_1}, G_D)$  satisfy*

$$DU_1 - U_1 D = D.$$

**Corollary 4.6.** *The up and down operators  $(D_1^*, U^*)$  of  $(G_{D_1}, G_U)$  satisfy*

$$U^* D_1^* - D_1^* U^* = U^*.$$

**4.2. Proof of Main results.** In this section, we prove Theorem 4.1, i.e.,

$$D(t)U(t') = \frac{1}{1-tt'}U(t')D(t),$$

and Theorem 4.2, i.e.,

$$D(t)U'(t') = (1+tt')U'(t')D(t).$$

First we rewrite these equations as the equations of the numbers of elements of some sets (Remark 4.10). We show the equation by constructing bijections (Lemmas 4.11 and 4.12).

**Lemma 4.7.** *The equation*

$$D(t)U(t') = \frac{1}{1-tt'}U(t')D(t)$$

*is equivalent to the equations*

$$(1) \quad D_j U_i = \sum_{k=0}^{\min(i,k)} U_{i-k} D_{j-k} \quad \text{for all } i, j.$$

**Lemma 4.8.** *The equation*

$$D(t)U'(t') = (1+tt')U'(t')D(t)$$

*is equivalent to the equations*

$$(2) \quad D_j U'_i = \sum_{k=0}^{\min(1,i,k)} U'_{i-k} D_{j-k} \quad \text{for all } i, j.$$

**Definition 4.9.** We respectively define sets  $N_{i,j}(T, T')$  and  $N'_{i,j}(T, T')$  of paths to be

$$\{ ((T, T''), (T', T'')) \in G_{U_i} \times G_{D_j} \}$$

and

$$\{ ((T, T''), (T', T'')) \in G_{U'_i} \times G_{D_j} \}.$$

We respectively define sets  $S_{j,i}(T, T')$  and  $S'_{j,i}(T, T')$  of paths to be

$$\{ ((T'', T), (T'', T')) \in G_{D_j} \times G_{U_i} \}$$

and

$$\{ ((T'', T), (T'', T')) \in G_{D_j} \times G_{U'_i} \}.$$

We also define  $\widetilde{S}_{j,i}(T, T')$  and  $\widetilde{S}'_{j,i}(T, T')$  by

$$\begin{aligned}\widetilde{S}_{j,i}(T, T') &= \prod_{k=0}^{\min(i,j)} S_{j-k,i-k}(T, T') \\ \widetilde{S}'_{j,i}(T, T') &= \prod_{k=0}^{\min(1,i,j)} S'_{j-k,i-k}(T, T'),\end{aligned}$$

where  $\coprod$  means the disjoint union.

*Remark 4.10.* By definition,

$$\begin{aligned}\langle D_j U_i T, T' \rangle &= |N_{i,j}(T, T')|, \\ \langle D'_j U_i T, T' \rangle &= |N'_{i,j}(T, T')|, \\ \langle U_j D_i T, T' \rangle &= |S'_{i,j}(T, T')|,\end{aligned}$$

and

$$\langle U_j D'_i T, T' \rangle = |S'_{i,j}(T, T')|$$

for each  $T, T' \in \mathbb{T}$ . Hence the equations (1) and (2) are respectively equivalent to the equations

$$|N_{i,j}(T, T')| = |\widetilde{S}_{j,i}(T, T')|$$

and

$$|N'_{i,j}(T, T')| = |\widetilde{S}'_{j,i}(T, T')|.$$

**Lemma 4.11.** *For each  $T, T' \in \mathbb{T}$  and each  $i, j \in \mathbb{N}$ , there exists bijection from  $N_{i,j}(T, T')$  to  $\widetilde{S}_{j,i}(T, T')$ .*

*Proof.* First we construct an element of  $\widetilde{S}_{j,i}(T, T')$  from an element of  $N_{i,j}(T, T')$ .

Let  $((T, T''), (T', T''))$  be an element of  $N_{i,j}(T, T')$ . Equivalently,  $(T, T'')$  is an edge of  $G_{U_i}$  such that  $T'' \ominus r_{T'',j} = T'$ . Let  $k$  be  $j - |r_{T'',j} \cap r_T|$ . We have  $r_{T,j-k} = r_{T'',j} \cap r_T$  since  $r_{T''}$  is one of the following:

$$\begin{aligned}r_T, \\ r_{T,l} \cup \{w_{T,l+1} 21^i \mid i \leq n\}, \\ r_{T,l} \cup \{w_{T,l+1} 1^i \mid i \leq n\}\end{aligned}$$

for some  $l, n \in \mathbb{N}$ . We consider

$$((T \ominus r_{T,j-k}, T), (T \ominus r_{T,j-k}, T')).$$

By definition,  $(T \ominus r_{T,j-k}, T)$  is in  $G_{D_{j-k}}$ . Since  $(T, T'')$  is in  $G_{U_i}$ ,  $(T \ominus r_{T,j-k}, T'' \ominus r_{T'',j})$  is in  $G_{U_{i-k}}$ . Since  $T' = T'' \ominus r_{T'',j}$ ,  $(T \ominus r_{T,j-k}, T')$  is in  $G_{U_{i-k}}$ . Hence we have  $((T, T \ominus r_{T,k}), (T', T \ominus r_{T,k})) \in \widetilde{S}_{j,i}(T, T')$ .

Next we construct an element of  $N_{i,j}(T, T')$  from an element of  $\tilde{S}_{j,i}(T, T')$ . Let  $((T''', T), (T''', T'))$  be an element of  $\tilde{S}_{j,i}(T, T')$ . Equivalently,  $(T''', T')$  is an edge of  $G_{U_{i-k}}$  such that  $T \ominus r_{T,j-k} = T'''$ .

First we consider the case where  $|r_T| > j - k$ . Let  $\omega = w_{T,j-k+1}$  and  $\omega' \in \nu_{T,j-k}^{-1}(\omega)$ . Since  $\omega' \in T \ominus r_{T,j-k}$  and  $\omega'2 \notin T \ominus r_{T,j-k}$ , we have

$$T'_{\omega'2} = \{ \omega'2, \omega'21, \dots, \omega'21^{n-1} \}$$

for some  $n \in \mathbb{N}$ , where this is empty for  $n = 0$ . For such  $n$ , let  $R$  denote

$$\{ \omega2, \omega21, \dots, \omega21^{n-1+k} \},$$

where this is empty for  $n + k = 0$ .

We define  $T''$  to be

$$\tilde{\nu}_{T,j-k}(T) \cup r_{T,j-k} \cup R.$$

Since  $r_{T''} = r_{T,j-k} \cup R$ ,  $((T, T''), (T', T''))$  is in  $N_{i,j}(T, T')$ .

Next we consider the case where  $|r_T| = j - k$ . Let  $\omega$  be

$$\max \{ w \notin r_T \mid w < w_{T,j-k} \}$$

and  $\omega' \in \nu_{T,j-k}^{-1}(r)$ . Since  $\omega' \in T \ominus r_{T,j-k}$  and  $\omega'2 \notin T \ominus r_{T,j-k}$ , we have

$$T'_{\omega'2} = \{ \omega'2, \omega'21, \dots, \omega'21^{n-1} \}$$

for some  $n \in \mathbb{N}$ , where this is empty for  $n = 0$ . For such  $n$ , let  $R$  denote

$$\{ w_{T,j-k+1}1, \dots, w_{T,j-k+1}1^{n-1+k} \},$$

where this is empty for  $n + k = 0$ .

We define  $T''$  to be

$$\tilde{\nu}_{T,j-k}(T) \cup r_{T,j-k} \cup R.$$

Since  $r_{T''} = r_{T,j-k} \cup R$ ,  $((T, T''), (T', T''))$  is in  $N_{i,j}(T, T')$ .

Thus we can construct an element of  $N_{i,j}(T, T')$  from an element of  $\tilde{S}_{j,i}(T, T')$ .

By the definition of them, these constructions are the inverses of each other. Hence we have the lemma.  $\square$

**Lemma 4.12.** *For  $T, T' \in \mathbb{T}$ , there exists a bijection from  $N'_{i,j}(T, T')$  to  $\tilde{S}'_{j,i}(T, T')$ .*

*Proof.* First we construct an element of  $\tilde{S}'_{j,i}(T, T')$  from an element of  $N'_{i,j}(T, T')$ .

Let  $((T, T''), (T', T''))$  be an element of  $N'_{i,j}(T, T')$ . Equivalently,  $(T, T'')$  is an edge of  $G_{U'_i}$  such that  $T'' \ominus r_{T'',j} = T'$ . Let  $k$  be  $|r_{T'',j} \cap r_T|$ . We have  $r_{T,j-k} = r_{T'',j} \cap r_T$  since  $r_{T''}$  is one of the following:

$$\begin{aligned} & r_T, \\ & r_{T,l} \cup \{ w_{T,l+1}12^n \}, \\ & r_{T,l} \cup \{ w_{T,l+1}2^n \} \end{aligned}$$

for some  $l, n \in \mathbb{N}$ . It also follows that  $k = 0$  or  $1$ . We consider

$$((T, T \ominus r_{T,k}), (T', T \ominus r_{T,k})).$$

By definition,  $(T \ominus r_{T,j-k}, T)$  is in  $G_{D_{j-k}}$ . Since  $(T, T'')$  is in  $G_{U'_i}$ ,  $(T \ominus r_{T,j-k}, T'' \ominus r_{T'',j})$  is in  $G_{U'_{i-k}}$ . Since  $T' = T'' \ominus r_{T'',j}$ ,  $(T \ominus r_{T,j-k}, T')$  is in  $G_{U'_{i-k}}$ . Hence we have  $((T, T \ominus r_{T,k}), (T', T \ominus r_{T,k})) \in \tilde{S}'_{j,i}(T, T')$ .

Next we construct an element of  $N'_{i,j}(T, T')$  from an element of  $\tilde{S}'_{j,i}(T, T')$ . Let  $((T''', T), (T''', T'))$  be an element of  $\tilde{S}'_{j,i}(T, T')$ . Equivalently,  $(T''', T')$  is an edge of  $G_{U'_{i-k}}$  such that  $T \ominus r_{T,j-k} = T'''$ .

First we consider the case where  $|r_T| > j - k$ . Let  $\omega = w_{T,j-k+1}$   $\omega' \in \nu_{T,j-k}^{-1}(\omega)$ . Since  $\omega' \in T \ominus r_{T,j-k}$  and  $\omega'2 \notin T \ominus r_{T,j-k}$ , we have

$$T'_{\omega'2} = \{ \omega'2, \dots, \omega'2^n \}$$

for some  $n \in \mathbb{N}$ , where this is empty for  $n = 0$ . For such  $n$ , let  $R$  denote

$$\{ \omega2, \dots, \omega2^{n+k} \},$$

where this is empty for  $n + k = 0$ .

We define  $T''$  to be

$$\tilde{\nu}_{T,j-k}(T) \cup r_{T,j-k} \cup R.$$

Since  $r_{T''} = r_{T,j-k} \cup \{ \omega2^{n+k} \}$ ,  $((T, T''), (T', T''))$  is in  $N'_{i,j}(T, T')$ .

Next we consider the case where  $|r_T| = j - k$ . Let  $\omega$  be

$$\max \{ w \notin r_T \mid w < w_{T,j-k} \}$$

and  $\omega' \in \nu_{T,j-k}^{-1}(\omega)$ . Since  $\omega' \in T \ominus r_{T,j-k}$  and  $\omega'2 \notin T \ominus r_{T,j-k}$ , we have

$$T'_{\omega'2} = \{ \omega'2, \dots, \omega'2^n \}$$

for some  $n \in \mathbb{N}$ , where this is empty for  $n = 0$ . For such  $n$ , let  $R$  denote

$$\{ w_{T,j-k}1, w_{T,j-k}12, \dots, w_{T,j-k}12^{n-1+k} \},$$

where this is empty for  $n + k = 0$ .

We define  $T''$  to be

$$\tilde{\nu}_{T,j-k}(T) \cup r_{T,j-k} \cup R.$$

Since

$$r_{T''} = r_{T,j-k} \cup \{ w_{T,j-k+1}12^{n-1+k} \},$$

$((T, T''), (T', T''))$  is in  $N'_{i,j}(T, T')$ .

Thus we can construct an element of  $N'_{i,j}(T, T')$  from an element of  $\tilde{S}'_{j,i}(T, T')$ .

By the definitions of them, these constructions are the inverses of each other. Hence we have the lemma.  $\square$

By Lemmas 4.11 and 4.12, we have Theorems 4.1 and 4.2.

## 5. APPLICATION

In this section, we consider a relation between our generalized Schur operators and the Loday-Ronco algebra.

*Remark 5.1.* We have correspondences between the sets  $N_{i,j}(T, T')$  and  $\tilde{S}_{i,j}(T, T')$  for all  $i, j$  by the proof of 4.11. From them, we can construct a Robinson-Schensted-Knuth correspondence for paths of  $G_U$  and  $G_D$  by the method in [4]. This correspondence is a generalization of the Loday-Ronco correspondence, which is a Robinson correspondence for binary trees.

*Remark 5.2.* Maxime Rey gave a construction of the Loday-Ronco algebra in [9]. He introduced a new Robinson-Schensted-Knuth correspondence for binary trees to construct the Loday-Ronco algebra. Some of our correspondences are equivalent to his correspondence.

**Definition 5.3.** For  $\lambda, \mu \in V$ , we define quasi-symmetric polynomials  $s_{\lambda, \mu}^D(t_1, \dots, t_n)$ ,  $s_U^{\lambda, \mu}(t_1, \dots, t_n)$  and  $s_{U'}^{\lambda, \mu}(t_1, \dots, t_n)$  by

$$\begin{aligned} s_{\lambda, \mu}^D(t_1, \dots, t_n) &= \langle D(t_1) \cdots D(t_n) T, T' \rangle \\ s_U^{\lambda, \mu}(t_1, \dots, t_n) &= \langle U(t_n) \cdots U(t_1) T', T \rangle \\ s_{U'}^{\lambda, \mu}(t_1, \dots, t_n) &= \langle U'(t_n) \cdots U'(t_1) T', T \rangle. \end{aligned}$$

*Remark 5.4.* For a labelling  $\varphi$  from  $T$  to  $\{1, \dots, m\}$ , we define  $t^\varphi = \prod_{w \in T} t_{\varphi(w)}$ . For a tree  $T$ , it follows by the definition of labellings that

$$\begin{aligned} s_{T, \emptyset}^D(t_1, \dots, t_n) &= \sum_{\psi} t^\psi, \\ s_U^{T, \emptyset}(t_1, \dots, t_n) &= \sum_{\varphi} t^\varphi, \\ s_{U'}^{T, \emptyset}(t_1, \dots, t_n) &= \sum_{\phi} t^\phi, \end{aligned}$$

where the first sum is over all binary-searching labellings  $\psi$  on  $T$ , the second sum is over all right-strictly-increasing labellings  $\varphi$  on  $T$ , and the last sum is over all left-strictly-increasing labellings  $\phi$  on  $T$ .

*Remark 5.5.* These polynomials  $s_U^{T, \emptyset}(t_1, \dots, t_n)$  and  $s_{T, \emptyset}^D(t_1, \dots, t_n)$  are the commutativizations of the basis elements  $\mathbf{Q}_T$  and  $\mathbf{P}_T$  of  $\mathbf{PBT}$  in Hivert-Novelli-Thibon [6].

*Remark 5.6.* Since  $D(t)$  and  $U(t)$  are generalized Schur operators, we have Pieri's formula for  $s_U^{T, \emptyset}(t_1, \dots, t_n)$  and  $s_{T, \emptyset}^D(t_1, \dots, t_n)$  by [7]. By [4], we have Cauchy identity for them. We also have a “skew” version of them.

We also have Pieri's formula and Cauchy identity for  $s_{U'}^{T, \emptyset}(t_1, \dots, t_n)$  and  $s_{T, \emptyset}^D(t_1, \dots, t_n)$

*Remark 5.7.* These polynomials are not symmetric in general. For example, since

$$\begin{aligned} D(t_1)D(t_2)\{0, 1, 12\} \\ &= D(t_1)(\{0, 1, 12\} + t_2\{0, 2\} + t_2^2\{0\}) \\ &= (\{0, 1, 12\} + t_1\{0, 2\} + t_1^2\{0\}) + t_2(\{0, 2\} + t_1\{0\}) + t_2^2(\{0\} + t_1\emptyset), \end{aligned}$$

$\langle D(t_n) \cdots D(t_1)T, \emptyset \rangle = t_1 t_2^2$  is not symmetric for  $T = \{0, 1, 12\}$ . The fact that  $D_i$  does not commute with  $D_j$  in general implies this fact.

## REFERENCES

- [1] S. Fomin, *Generalized Robinson-Schensted-Knuth correspondence*, Zariski Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **155** (1986), 156–175, 195 (Russian); English transl., J. Soviet Math. 41(1988), 979–991.
- [2] S. Fomin, *Duality of graded graphs*, J. Algebraic Combin. **3** (1994), 357–404.
- [3] S. Fomin, *Schensted algorithms for dual graded graphs*, J. Algebraic Combin. **4** (1995), 5–45.
- [4] S. Fomin, *Schur operators and Knuth correspondences*, J. Combin. Theory, Ser. A **72** (1995), 277–292.
- [5] Ira M. Gessel, *Counting paths in Young’s lattice*, J. Statistical planning and inference. **34** (1993), 125–134.
- [6] F. Hivert, J. Novelli and J. Thibon, *The algebra of binary search trees*, Theor. Comput. Sci. 339, 1 (Jun. 2005), 129–165. DOI=<http://dx.doi.org/10.1016/j.tcs.2005.01.012>
- [7] Y. Numata, *Pieri’s Formula for Generalized Schur Polynomials*, preprint, arXiv:math.CO/0606386.
- [8] J. Nzeutchap, Graded Graphs and Fomin’s  $r$ -correspondences associated to the Hopf Algebras of Planar Binary Trees, Quasi-symmetric Functions and Noncommutative Symmetric Functions, FPSAC ’06, 2006. <http://garsia.math.yorku.ca/fpsac06/papers/53.pdf>
- [9] M. Rey, A new construction of the Loday-Ronco algebra, FPSAC ’06, 2006. <http://garsia.math.yorku.ca/fpsac06/papers/51.pdf>
- [10] T. Roby, *Applications and extensions of Fomin’s generalization of the Robinson-Schensted correspondence to differential posets*, Ph.D. thesis, M.I.T., 1991.
- [11] R. Stanley, *Differential posets*, J. American Math. Soc, **1** (1988), 919–961.
- [12] R. Stanley, *Variations on differential posets*, Invariant theory and tableaux (Stanton, D., ed.), IMA volumes in mathematics and its applications, Springer-Verlag, New York, 145–165.

KITA 10, NISHI 8, KITA-KU, SAPPORO, HOKKAIDO, 060-0810, JAPAN.

*E-mail address:* nu@math.sci.hokudai.ac.jp